BIHARMONIC ISOMETRIC IMMERSIONS INTO AND BIHARMONIC RIEMANNIAN SUBMERSIONS FROM BERGER 3-SPHERES

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Abstract

In this paper, we study biharmonic isometric immersions of a surface into and biharmonic Riemannian submersions from 3-dimensional Berger spheres. We obtain a classification of proper biharmonic isometric immersions of a surface with constant mean curvature into Berger 3-spheres. We also give a complete classification of proper biharmonic Hopf tori in Berger 3-sphere. For Riemannian submersions, we prove that a Riemannian submersion from Berger 3-spheres into a surface is biharmonic if and only if it is harmonic.

1. Introduction and preliminaries

In this paper, we work in the category of smooth objects, so manifolds, maps, vector fields, etc, are assumed to be smooth unless it is stated otherwise.

Recall a harmonic map $\varphi : (M, g) \to (N, h)$ of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) that if $\varphi|_{\Omega}$ is a critical point of the energy functional defined by

$$E(\varphi, \Omega) = \frac{1}{2}\int_{\Omega} |\mathrm{d}\varphi|^2 \,\mathrm{d}x.$$

The Euler-Lagrange equation (see [2, 14]) is given by the vanishing of the tension field $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$, i.e., $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi = 0$.
In 1983, J. Eells and L. Lemaire [14] extended the notion of harmonic maps to

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biharmonic maps which are critical points of the bienergy functional

$$E^2(\varphi, \Omega) = \frac{1}{2}\int_\Omega |\tau(\varphi)|^2 \, \mathrm{d}x,$$

for every compact subset Ω of M, where $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ is the tension field of φ. In 1986, G.Y. Jiang [17] first computed the first variation of the functional, and obtained that φ is biharmonic if and only if its bitension field vanishes identically, i.e.,

$$\tau^2(\varphi) := \mathrm{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0,$$

where R^N is the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

Naturally, any harmonic map is always biharmonic.

A Riemannian submersion is called a **biharmonic Riemannian submersion** if the Riemannian submersion is a biharmonic map. Similarly, a submanifold is called a biharmonic submanifold if the isometric immersion that defines the submanifold is a biharmonic map. As is well known, an isometric immersion is harmonic if and only if it is minimal, and hence biharmonic submanifolds include minimal submanifolds as a subset. We use **proper biharmonic maps (respectively, Riemannian submersion, isometric immersion, submanifold)** to name those biharmonic maps (respectively, Riemannian submersion, isometric immersion, submanifold) which are not harmonic.

Many recent works in the geometric study of biharmonic maps have been focused on the existence of a proper biharmonic map between two "good" model spaces. The so-called "good" model spaces include space forms, more general symmetric, homogeneous spaces, etc. It would be also important to classify all proper biharmonic maps between two model spaces where the existence is known. We refer to two classification problems as follows

Chen's conjecture [12, 13, 11]: every biharmonic submanifold in a Euclidean space $\mathbb{R}^n$ is minimal (i.e., harmonic)

The generalized Chen's conjecture: every biharmonic submanifold of a Riemannian manifold of non positive curvature must be harmonic (minimal) (see e.g., [4–13]).

The Chen's conjecture is still open for the general case, and some results for affirmative answers to Chen's conjecture were shown in [19, 7, 26, 28, 15]. For the generalized Chen's conjecture, Ou and Tang ([27]) gave many counter examples in a Riemannian manifold of negative curvature. For some recent progress on biharmonic submanifolds, we refer the readers to [1], [4-13], [23-30], etc., and the

references therein.

On the other hand, as it is well known that Riemannian submersions can be considered as the dual notion of isometric immersions (i.e., submanifolds), it is very interesting to study biharmonicity of Riemannian submersions between Riemannian manifolds. In 2002, Oniciuc [20] first studied biharmonic Riemannian submersions. In 2010, Wang and Ou [33] first used the so-called integrability data to study biharmonicity of a Riemannian submersion from a generic 3-manifold, they then used the main tool to derived a complete classification of biharmonic Riemannian submersions from a 3-dimensional space form into a surface. In a recent paper [1], Akyol and Ou studied biharmonicity of a general Riemannian submersion and obtained biharmonic equations for Riemannian submersions with one-dimensional fibers and Riemannian submersions with basic mean curvature vector fields of fibers. In particular, the authors of [1] used the so-called integrability data to study biharmonic Riemannian submersions from $(n+1)$-dimensional spaces with one-dimensional fibers and obtained many examples of biharmonic Riemannian submersions. In [30], the author studied biharmonicity a more general setting of Riemannian submersions with a S^1 fiber over a compact Riemannian manifold. In 2018, the authors in [16] studied generalized harmonic morphisms and obtained many examples of biharmonic Riemannian submersions which are maps between Riemannian manifolds that pull back local harmonic functions to local biharmonic functions.

Finally, we refer an interested reader to the recent works [25] and [34] for complete classifications of constant mean curvature proper biharmonic surfaces in Thurston's 3-dimensional geometries and in BCV 3-spaces, a complete classification of proper biharmonic Hopf cylinders BCV 3-spaces, complete classification of proper biharmonic Riemannian submersions from BCV 3-diemnsional spaces into a surface, and some constructions of examples of proper biharmonic Riemannian submersion from $H^2\times\mathbb{R}\to\mathbb{R}^2$, or, $\widetilde{SL}(2,\mathbb{R})\to\mathbb{R}^2$.

In this paper, we will study biharmonic isometric immersions of a surface into and biharmonic Riemannian submersions from 3-dimensional Berger sphere S^3_ε. We show that an isometric immersion of a surface with constant mean curvature into Berger 3-sphere is proper biharmonc if and only if the surface is a part of $S^2(1/\sqrt{2})$ in S^3 or a part of a Hopf torus in S^3_ε whose base curve is a circle with radius $r=1/\sqrt{8-4\varepsilon^2}$ in the base sphere $S^2(\frac{1}{2})$. We also give a complete classification of proper biharmonic Hopf tori in a Berger 3-sphere. For Riemannian

submersions, we prove that a Riemannian submersion from a Berger 3-sphere into a surface is biharmonic if and only if it is harmonic.

2. Biharmonic isometric immersions of a surface with constant mean curvature into Berger 3-sphere S^3_ε

Biharmonic surfaces in 3-dimensional space forms have been completely classified in [18], [11], [9], [10]), and also biharmonic constant mean curvature surfaces in 3-dimensional BCV spaces and Sol space have been completely classified ([25]). In this section, we obtain a complete classification of isometric immersions of a surface with constant mean curvature into a Berger 3-sphere S^3_ε. We also derive a complete classification of proper biharmonic Hopf cylinders in a Berger 3-sphere.

Let us recall the definition of the so-called 3-dimensional Berger sphere (see e.g., [4]). Consider the Hopf map $\psi : S^3(1) \to S^2(\frac{1}{2})$ given by

(1)
$$\psi(x^1, x^2, x^3, x^4) = \tfrac{1}{2}(2x^1x^3 + 2x^2x^4, 2x^2x^3 - 2x^1x^4, (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2),$$

or

$$\psi(z, w) = \tfrac{1}{2}(2zw, |z|^2 - |w|^2), \tag{2}$$

where $z = x^1 + ix^2$, $w = x^3 + ix^4$ and $S^2(\frac{1}{2})$ denotes a 2-sphere with radius $\frac{1}{2}$ (i.e., constant Gauss curvature 4). It is not difficult to see that the map ψ is a Riemannian submersion with totally geodesic fibers $\psi^{-1}(\psi(z, w))$ which are the great circle passing through (z, w) and (iz, iw).

With respect to the Hopf fibration, the following deformation of the standard metric g on S^3 gives a family of metric on the sphere:

$$g_\varepsilon|_{T^HS^3\times T^HS^3} = g|_{T^HS^3\times T^HS^3}, g_\varepsilon|_{T^VS^3\times T^VS^3} = \varepsilon^2 g, g_\varepsilon|_{T^HS^3\times T^VS^3} = 0, \tag{3}$$

where T^VS^3 and T^HS^3 denote respectively the vertical and the horizontal spaces determined by ψ. We call a sphere a Berger 3-sphere if the sphere S^3 endowed with the metric g_ε. A Berger 3-sphere is denoted by S^3_ε, i.e., $S^3_\varepsilon = (S^3, g_\varepsilon)$, where $\varepsilon \neq 0$. Suppose $x \in S^3$, we have the following facts:

(i) the vector fields

$$\begin{aligned} &X_1(x) = (-x^2, x^1, -x^4, x^3),\ X_2(x) = (-x^4, -x^3, x^2, x^1),\\ &X_3(x) = (-x^3, x^4, x^1, -x^2) \end{aligned} \tag{4}$$

parallelize S^3,

(ii) X_1 is tangent to the fibres of the Hopf map (i.e. $d\psi(X_1) = 0$), and

(iii)X_2 and X_3 are horizontal, but not basic.

From (3) we have a global orthonormal frame field

$$\{E_1 = X_2, E_2 = X_3, E_3 = \varepsilon^{-1}X_1\} \tag{5}$$

on S^3_ε.

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{6}$$

and the Riemannian and the Ricci curvatures:

$$\begin{aligned} &R(X,Y,Z,W) = g(R(Z,W)Y,X),\\ &\mathrm{Ric}(X,Y) = \mathrm{Trace}_g R = \sum_{i=1}^{3} R(Y,e_i,X,e_i) = \sum_{i=1}^{3}\langle R(X,e_i)e_i,Y\rangle. \end{aligned} \tag{7}$$

With respect to the frame, a straightforward computation shows that

$$[E_1,E_2] = 2\varepsilon E_3,\ \ [E_2,E_3] = \frac{2}{\varepsilon}E_1,\ [E_3,E_1] = \frac{2}{\varepsilon}E_2. \tag{8}$$

The Levi-Civita connection of the metric g_ε has the expression as

$$\begin{cases} \nabla_{E_1}E_1 = 0,\ \ \nabla_{E_1}E_2 = \varepsilon E_3,\ \ \nabla_{E_1}E_3 = -\varepsilon E_2,\\ \nabla_{E_2}E_1 = -\varepsilon E_3,\ \ \nabla_{E_2}E_2 = 0,\ \ \nabla_{E_2}E_3 = \varepsilon E_1,\\ \nabla_{E_3}E_1 = \frac{2-\varepsilon^2}{\varepsilon}E_2,\ \ \nabla_{E_3}E_2 = -\frac{2-\varepsilon^2}{\varepsilon}E_1,\ \ \nabla_{E_3}E_3 = 0. \end{cases} \tag{9}$$

A further computation (see also [4]) gives the possible nonzero components of the curvatures:

$$\begin{aligned} &R_{1212} = g(R(E_1,E_2)E_2,E_1) = 4-3\varepsilon^2,\\ &R_{1313} = g(R(E_1,E_3)E_3,E_1) = R_{2323} = g(R(E_2,E_3)E_3,E_2) = \varepsilon^2,\\ &\text{all other } R_{ijkl} = g(R(E_k,E_l)E_j,E_i) = 0,\ i,j,k,l = 1,2,3. \end{aligned} \tag{10}$$

and the Ricci curvature:

$$\begin{aligned} &\mathrm{Ric}\,(E_1,E_1) = \mathrm{Ric}\,(E_2,E_2) = 4-2\varepsilon^2,\\ &\mathrm{Ric}\,(E_3,E_3) = 2\varepsilon^2,\ \text{all other } \mathrm{Ric}\,(E_i,E_j) = 0,\ i\neq j. \end{aligned} \tag{11}$$

Remark 1. From (i), (ii), (iii), (5), (8) and (9), we would like to point out the following:
(a): The map $\psi : S^3_\varepsilon \to S^2(\frac{1}{2})$, $\psi(z,w) = \frac{1}{2}(2zw, |z|^2-|w|^2)$, where $z = x^1+ix^2$, $w = x^3+ix^4$, is a Riemannian submersion with totally geodesic from a Berger 3-sphere S^3_ε to a 2-sphere $S^2(\frac{1}{2})$ with constant Gauss curvature 4, i.e., the Riemannian submersion is harmonic.
(b): $\{E_1 = X_2, E_2 = X_3, E_3 = \varepsilon^{-1}X_1\}$ is an orthonormal frame on S^3_ε with E_3 being vertical.

(c): $\{E_1 = X_2, E_2 = X_3\}$ is horizontal, but not basic.

We will use the following equation for biharmonic hypersurfaces in a generic Riemannian manifold.

Theorem 2.1. ([24]) *Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta = H\xi$. Then φ is biharmonic if and only if:*

$$\begin{cases}\Delta H - H|A|^2 + H\mathrm{Ric}^N(\xi,\xi) = 0,\\ 2A\,(\mathrm{grad}\, H) + \frac{m}{2}\mathrm{grad}\, H^2 - 2\,H\,(\mathrm{Ric}^N\,(\xi))^{\top} = 0,\end{cases} \tag{12}$$

where $\mathrm{Ric}^N : T_qN \longrightarrow T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \mathrm{Ric}^N\,(Z), W\rangle = \mathrm{Ric}^N(Z,W)$ and A is the shape operator of the hypersurface with respect to the unit normal vector ξ.

We now study biharmonic constant mean curvature (CMC) surfaces in a 3-dimensional Berger sphere S^3_ε.

Theorem 2.2. *A constant mean curvature surface in 3-dimensional Berger spheres S^3_ε is proper biharmonic if and only if it is a part of:*
(i) $S^2(1/\sqrt{2})$ in S^3, or
(ii) a Hopf torus in S^3_ε, i.e., the inverse image of the Hopf fibration of a circle of radius $r = \frac{1}{2\sqrt{2-\varepsilon^2}}$ with $\varepsilon^2 < 1$ in the base sphere $S^2(\frac{1}{2})$.

Proof. Let $\{e_1 = \sum\limits_{i=1}^{3} a_1^i E_i, \quad e_2 = \sum\limits_{i=1}^{3} a_2^i E_i, \quad \xi = \sum\limits_{i=1}^{3} a_3^i E_i\}$ be an orthonormal frame on S^3_ε adapted to the surface with ξ being normal. We then use the Ricci curvature (11) to have $\mathrm{Ric}\,(\xi,\xi) = 4 - 2\varepsilon^2 + (4\varepsilon^2 - 4)(a_3^3)^2$, $(\mathrm{Ric}\,(\xi))^{\top} = (4\varepsilon^2 - 4)a_1^3 a_3^3 e_1 + (4\varepsilon^2 - 4)a_2^3 a_3^3 e_2$. From these and the biharmonic surface (12), we conlude that a surface with constant mean curvature H is biharmonic if and only if

$$\begin{cases}-H\,[|A|^2 - (4 - 2\varepsilon^2) - (4\varepsilon^2 - 4)(a_3^3)^2] = 0,\\ (4\varepsilon^2 - 4)a_1^3 a_3^3 H = 0,\\ (4\varepsilon^2 - 4)a_2^3 a_3^3 H = 0,\end{cases} \tag{13}$$

which has solution $H = 0$ implying that the surface is harmonic (minimal), or,

$$\begin{cases}|A|^2 - (4 - 2\varepsilon^2) - (4\varepsilon^2 - 4)(a_3^3)^2 = 0,\\ (4\varepsilon^2 - 4)a_1^3 a_3^3 = 0,\\ (4\varepsilon^2 - 4)a_2^3 a_3^3 = 0.\end{cases} \tag{14}$$

We solve (14) by considering the following cases:
Case I: $4\varepsilon^2-4=0$. In this case, we have $|A|^2=2$ and the corresponding Berger sphere S^3_ε is a standard 3-dimensional sphere S^3. It follows from [9] [10] that the only proper biharmonic surface in a 3-dimensional sphere S^3 is a part of $S^2(1/\sqrt{2})$ in S^3 .
Case II: $4\varepsilon^2-4\neq 0$. In this case, by the last two equations of (14), we have either $a_3^3=0$ or $a_1^3=a_2^3=0$.
For Case II-A: $a_3^3=0$, using the first equation of (14) we have

$$|A|^2=4-2\varepsilon^2. \tag{15}$$

Noting that $a_3^3=0$ implies that the normal vector field of the surface Σ is always orthogonal to $E_3=\varepsilon^{-1}X_1$ so we can choose an another orthonormal frame $\{e_1=aE_1+bE_2, e_2=E_3, \xi=bE_1-aE_2\}$ adapted to the surface with $a^2+b^2=1$ and ξ being the unit normal vector filed. We use (9) to compute

$$\nabla_{e_1}\xi=\{ae_1(b)-be_1(a)\}e_1-\varepsilon e_2,\ \nabla_{e_2}\xi=\left(ae_2(b)-be_2(a)+\frac{2-\varepsilon^2}{\varepsilon}\right)e_1. \tag{16}$$

With respect to the chosen adapted orthonormal frame, by a further computation, the second fundamental form of the surface Σ given by
(17)

$$\begin{aligned} &h(e_1,e_1)=-\langle\nabla_{e_1}\xi,e_1\rangle=-ae_1(b)+be_1(a),\ h(e_1,e_2)=-\langle\nabla_{e_1}\xi,e_2\rangle=\varepsilon,\\ &h(e_2,e_1)=-\langle\nabla_{e_2}\xi,e_1\rangle=-ae_2(b)+be_2(a)-\tfrac{2-\varepsilon^2}{\varepsilon},\ h(e_2,e_2)=-\langle\nabla_{e_2}\xi,e_2\rangle=0.\end{aligned}$$

By (16) and (17), we have

$$\nabla_{e_1}e_1=-\{ae_1(b)-be_1(a)\}\xi=2H\xi,\ \nabla_{e_1}\xi=-2He_1-\varepsilon e_2,\ \nabla_{e_1}e_2=\varepsilon\xi. \tag{18}$$

It follows from (17), the symmetry $h(e_1,e_2)=h(e_2,e_1)$, and $0=e_2(a^2+b^2)=2ae_2(a)+2be_2(b)$ that $e_2(a)=\frac{2}{\varepsilon}b, e_2(b)=-\frac{2}{\varepsilon}a$.
Denoting by $\alpha_1=\xi=bE_1-aE_2, \alpha_2=e_1=aE_1+bE_2$, and $\alpha_3=e_2=E_3$, a straightforward computation using (8) and (18) gives
(19)

$$[\alpha_1,\alpha_3]=0,\ [\alpha_2,\alpha_3]=0,\ [\alpha_1,\alpha_2]=(b\alpha_1(a)-a\alpha_1(b))\,\alpha_1+2H\alpha_2+2\varepsilon\alpha_3.$$

By Remark 1, the map $\psi:S^3_\varepsilon\to S^2(\frac{1}{2})$

$$\psi(x^1,x^2,x^3,x^4)=\tfrac{1}{2}(2x^1x^3+2x^2x^4,2x^2x^3-2x^1x^4,(x^1)^2+(x^2)^2-(x^3)^2-(x^4)^2),$$

or

$$\psi(z,w)=\tfrac{1}{2}(2zw,|z|^2-|w|^2),$$

is a Riemannian submersion with totally geodesic fibers with an orthonoromal frame $\{E_1,E_2,\ \alpha_3=E_3\}$ on S^3_ε with E_3 being vertical, where $z=x^1+ix^2, w=x^3+ix^4$.

An interesting thing is that, By (19), one see that the orthonormal frame $\{e_1, e_2, \xi\}$ adapted to the cmc surface happens to be an orthonormal frame $\{\alpha_1 = \xi = bE_1 - aE_2, \alpha_2 = e_1 = aE_1 + bE_2, \alpha_3 = e_2 = E_3\}$ adapted to the Riemannian submersion ψ. The tangent vector field of the surface $e_2 = E_3$ also implies that the surface contains all the fibers of ψ which intersect the surface. Thus, locally, the surface is formed by the fibers of ψ that pass through every point on an integral curve of $e_1 = \alpha_2$ which is horizontal and basic to the Riemannian submersion ψ. It follows that the cmc surface is actually a Hopf torus, i.e., the inverse image of a curve on base sphere of the Hopf fibration.

More precisely, we can determine the torus as follows.

Let $\gamma : I \to S^3_\varepsilon$, $\gamma = \gamma(s)$, be an integral curve along the basic vector field $\alpha_2 = e_1 = aE_1 + bE_2$ on the surface Σ with arclength parameter, then it is horizontal with respect to the Riemannian submersion ψ. Let $\beta(s) = \psi(\gamma(s))$ be the curve in the base space of the Riemannian submersion, then the surface Σ can be viewed as $\Sigma = \cup_{s\in I}\psi^{-1}(\beta(s))$, a Hopf torus over the curve $\beta(s) \subset S^2(\frac{1}{2})$.

Noting also that $\alpha_1 = \xi = bE_1 - aE_2, \alpha_2 = e_1 = aE_1 + bE_2, \alpha_3 = e_2 = E_3$, then (18) turns into

$$\begin{aligned} &\nabla_{\alpha_2}\alpha_2 = 2H\alpha_1 = k_g\alpha_1,\ \nabla_{\alpha_2}\alpha_1 = -2H\alpha_2 - \varepsilon\alpha_3 = -k_g\alpha_2 + \tau_g\alpha_3,\\ &\nabla_{\alpha_2}\alpha_3 = \varepsilon\alpha_1 = -\tau_g\alpha_1, \end{aligned} \tag{20}$$

which is the Frenet formula of the curve $\gamma = \gamma(s)$ (see also [[32], Example 3.4.1], and means that k_g is the geodesic curvature of the base curve, τ_g is the geodesic torsion of $\gamma(s)$. It follows from Eqs. (17) and (20) that

$$|A|^2 = k_g^2 + \varepsilon^2,\ H = k_g/2. \tag{21}$$

Comparing (15) and (21)) we get

$$k_g^2 = 4(1-\varepsilon^2) > 0,\ H^2 = 1 - \varepsilon^2 > 0.$$

Since the curve in the base sphere $S^2(\frac{1}{2})$ has constant geodesic curvature and hence $k_g = 2\sqrt{1-\varepsilon^2}$, one can check that this curve, considered as a curve in Euclidean 3-space of which $S^2(\frac{1}{2})$ is a subset, has curvature $k = \sqrt{k_g^2 + k_n^2} = 2\sqrt{2-\varepsilon^2}$ and torsion $\tau = -\frac{2k'}{kk_g} = 0$. Combining this, we find the base curve of the Hopf cylinder to be a circle on $S^2(\frac{1}{2})$ with radius $r = \frac{1}{2\sqrt{2-\varepsilon^2}}$.

For Case II-B: $a_1^3 = a_2^3 = 0$ and $a_3^3 = \pm 1$. It follows, in this case, Span$\{e_1, e_2\}$ = Span$\{E_1, E_2\}$. This implies the distribution determined by $\{E_1, E_2\}$ is integrable and hence (by Frobenius theorem) is involutive. This leads to $\varepsilon = 0$ by (8), a contradiction.

Summarizing all results proved above we obtain the theorem.

□

Theorem 2.3. *Let $\psi : S^3_\varepsilon \to S^2(\frac{1}{2})$, $\psi(z,w) = \frac{1}{2}(2zw, |z|^2 - |w|^2)$ be the Hopf fibration, and $\beta : I \to S^2(\frac{1}{2})$ be an immersed regular curve parameterized by arc length. Then the Hopf torus $\Sigma = \cup_{s\in I}\psi^{-1}(\beta(s))$ is a proper biharmonic surface in a Berger 3-sphere S^3_ε if and only if it is the curve $\beta(s)$ on the base sphere $S^2(\frac{1}{2})$ is circle of radius $r = \frac{1}{2\sqrt{2-\varepsilon^2}}$ with $\varepsilon^2 < 1$.*

Proof. Let $\beta : I \to S^2(\frac{1}{2})$ be an immersed regular curve parameterized by arc length with the geodesic curvature k_g. It follows from a result in [24] that we can take the horizontal lifts of the tangent and the principal normal vectors of the curve β : $X = aE_1 + bE_2$ and $\xi = bE_1 - aE_2$ (where $a^2 + b^2 = 1$) together with $V = E_3$ to be an adapted orthonormal frame of the Hopf cylinder. A direct computation using (11) gives:

$$\mathrm{Ric}\,(\xi,\xi) = 4 - 2\varepsilon^2,\ \mathrm{Ric}\,(\xi, X) = \mathrm{Ric}\,(\xi, V) = 0. \tag{22}$$

We can check that the the geodesic torsion of the lifting curve $\psi^{-1}(\beta(s))$

$$\tau_g = -\langle \nabla_X V, \xi\rangle = -\langle \nabla_{aE_1+bE_2} E_3, bE_1 - aE_2\rangle = -\varepsilon. \tag{23}$$

It follows from Eq. (16) in [24] that the surface $\sum$ in S^3_ε is biharmonic if and only if

$$\begin{cases} k_g'' - k_g(k_g^2 + 2\tau_g^2) + k_g\mathrm{Ric}(\xi,\xi) = 0,\\ 3k_g'k_g - 2k_g\mathrm{Ric}(\xi, X) = 0,\\ k_g'\tau_g + k\mathrm{Ric}(\xi, V) = 0.\end{cases}$$

Substituting (22) and (23) into the above equation, we get

$$k_g'' - k_g\left(k_g^2 - (4 - 4\varepsilon^2)\right) = 0,\ 3k_g'k_g = 0 \text{ and } -\varepsilon k_g' = 0. \tag{24}$$

We solve (24)to have $k_g = 0$, which means that the surface Σ is minimal surface, or β has constant geodesic curvature $k_g^2 = 4 - 4\varepsilon^2 > 0$. It is easy to see from [24] (Page 229) that the mean curvature of the Hopf cylinder is given by $H = \frac{k_g}{2}$ and $|A|^2 = k_g^2 + 2\tau_g^2 = 4 - 2\varepsilon^2 = \mathit{constant}$. From these we deduce that the Hopf cylinder $\Sigma = \cup_{s\in I}\psi^{-1}(\beta(s))$ is proper biharmonic if and only if

$$H^2 = 1 - \varepsilon^2 > 0,\ |A|^2 = 4 - 2\varepsilon^2 > 0. \tag{25}$$

We apply the characterizations of Hopf cylinders in S^3_ε given in Theorem 2.2 to obtain the Theorem.

□

Corollary 2.4. *A totally umbilical surface in a Berger 3-sphere S^3_ε is proper biharmonic if and only if it is a part of $S^2(1/\sqrt{2})$ in S^3.*

Proof. It follows from a result in [25] that a totally umbilical biharmonic surface in 3-dimensional Riemannian manifolds must have constant mean curvature H. This, together with Theorem 2.2, implies that the only potential totally umbilical proper biharmonic surface is a part of $S^2(1/\sqrt{2})$ in S^3. □

Since $\varepsilon^2 = 1$, we see that a potential Berger 3-sphere S^3_ε has to be 3-sphere S^3. Applying Corollary 2.4, we get

Corollary 2.5. *A totally umbilical surface in a Berger 3-sphere S^3_ε with $\varepsilon^2 \neq 1$ is biharmonic if and only if it is minimal.*

3. Biharmonic Riemannian submersions from a Berger 3-sphere S^3_ε

As Riemannian submersions can be considered as the dual notion of isometric immersions, it would be interesting to study biharmonic Riemannian submersions. In a recent paper [34], the authors classified all proper Riemannian submersions from BCV 3-diemnsional spaces into a surface, and proved that a proper biharmonic Riemannian submersion from a BCV 3-diemnsional space exists only in $H^2\times\mathbb{R}\to\mathbb{R}^2$, or, $\widetilde{SL}(2,\mathbb{R})\to\mathbb{R}^2$ of which some examples were given. In this section, we give a complete classification of biharmonic Riemannian submersions from a Berger 3-sphere into a surface.

Let $\pi : S^3_\varepsilon \to (N^2, h)$ be a Riemannian submersion from Berger 3-sphere S^3_ε with an orthonormal frame $\{e_1,\ e_2,\ e_3\}$ and e_3 being vertical. Then, we have the following (26)-(32)(see [34] for details)

(26)
$$[e_1,e_3]=f_3e_2+\kappa_1e_3,[e_2,e_3]=-f_3e_1+\kappa_2e_3,[e_1,e_2]=f_1e_1+f_2e_2-2\sigma e_3,$$

where $\{f_1,f_2,f_3,\ \kappa_1,\ \kappa_2,\ \sigma\}$ is the (generalized) integrability data of the Riemannian submersion π. The frame $\{e_1,\ e_2,\ e_3\}$ is adapted to Riemannian submersion if and only if $f_3=0$ holds, and hence $\{f_1,f_2,\kappa_1,\ \kappa_2,\ \sigma\}$ is called the integrability data of the adapted frame of the Riemannian submersion π.
The Levi-Civita connection for the frame $\{e_1,\ e_2,\ e_3\}$ given by
(27)
$$\begin{aligned}
&\nabla_{e_1}e_1=-f_1e_2,\ \ \nabla_{e_1}e_2=f_1e_1-\sigma e_3,\ \ \nabla_{e_1}e_3=\sigma e_2,\\
&\nabla_{e_2}e_1=-f_2e_2+\sigma e_3,\ \ \nabla_{e_2}e_2=f_2e_1,\ \ \nabla_{e_2}e_3=-\sigma e_1,\\
&\nabla_{e_3}e_1=-\kappa_1e_3+(\sigma-f_3)e_2,\nabla_{e_3}e_2=-(\sigma-f_3)e_1-\kappa_2e_3,\nabla_{e_3}e_3=\kappa_1e_1+\kappa_2e_2,
\end{aligned}$$

the Jacobi identities as

$$\begin{cases} e_3(f_1)+(\kappa_1+f_2)f_3-e_1(f_3)=0,\\ e_3(f_2)+(\kappa_2-f_1)f_3-e_2(f_3)=0,\\ 2e_3(\sigma)+\kappa_1 f_1+\kappa_2 f_2+e_2(\kappa_1)-e_1(\kappa_2)=0, \end{cases} \tag{28}$$

and if denoting by $e_i=\sum_{j=1}^{3}a_i^jE_j,\ i=1,2,3,$ using (9), (10) and (27), then we have

(29)

$$\begin{cases} R^M(e_1,e_3,e_1,e_2)=-e_1(\sigma)+2\kappa_1\sigma=-a_2^3a_3^3R,\\ R^M(e_1,e_3,e_1,e_3)=e_1(\kappa_1)+\sigma^2-\kappa_1^2+\kappa_2f_1=(a_2^3)^2R+\varepsilon^2,\\ R^M(e_1,e_3,e_2,e_3)=e_1(\kappa_2)-e_3(\sigma)-\kappa_1f_1-\kappa_1\kappa_2=-a_1^3a_2^3R,\\ R^M(e_1,e_2,e_1,e_2)=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2+2f_3\sigma-3\sigma^2=(a_3^3)^2R+\varepsilon^2,\\ R^M(e_1,e_2,e_2,e_3)=-e_2(\sigma)+2\kappa_2\sigma=a_1^3a_3^3R,\\ R^M(e_2,e_3,e_1,e_3)=e_2(\kappa_1)+e_3(\sigma)+\kappa_2f_2-\kappa_1\kappa_2=-a_1^3a_2^3R,\\ R^M(e_2,e_3,e_2,e_3)=\sigma^2+e_2(\kappa_2)-\kappa_1f_2-\kappa_2^2=(a_1^3)^2R+\varepsilon^2, \end{cases}$$

where $R=4-4\varepsilon^2$.
Gauss curvature of the base space is given by

$$K^N=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2+2f_3\sigma. \tag{30}$$

Clearly,

$$e_3(K^N)=e_3\{e_1(f_2)-e_2(f_1)-f_1^2-f_2^2+2f_3\sigma\}=0, \tag{31}$$

when $f_3=0$, then Gauss curvature of the base space turns into

$$K^N=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2. \tag{32}$$

We state biharmonic equation for Riemannian submersion from 3-manifolds which will be later used in the rest of this paper.

Lemma 3.1. *([33]) Let $\pi:(M^3,g)\to(N^2,h)$ be a Riemannian submersion with the adapted frame $\{e_1,\ e_2,\ e_3\}$ and the integrability data $\{f_1,f_2,\kappa_1,\ \kappa_2,\ \sigma\}$.*

Then, the Riemannian submersion π is biharmonic if and only if

(33)
$$\begin{cases}-\Delta^M\kappa_1 - 2\sum\limits_{i=1}^{2} f_i e_i(\kappa_2) - \kappa_2\sum\limits_{i=1}^{2}(e_i(f_i) - \kappa_i f_i) + \kappa_1\left(-K^N + \sum\limits_{i=1}^{2} f_i^2\right) = 0,\\ -\Delta^M\kappa_2 + 2\sum\limits_{i=1}^{2} f_i e_i(\kappa_1) + \kappa_1\sum\limits_{i=1}^{2}(e_i(f_i) - \kappa_i f_i) + \kappa_2\left(-K^N + \sum\limits_{i=1}^{2} f_i^2\right) = 0,\end{cases}$$

where $K^N = R^N_{1212}\circ\pi = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2$ is Gauss curvature of Riemannian manifold (N^2, h).

Proposition 3.2. *(see [34]) Let $\pi : (M^3, g) \longrightarrow (N^2, h)$ be a Riemannian submersion from 3-manifolds with an orthonormal frame $\{e_1, e_2, e_3\}$ and e_3 being vertical. If $\nabla_{e_1}e_1 = 0$, then either $\nabla_{e_2}e_2 = 0$; or, $\nabla_{e_2}e_2 \not\equiv 0$, and the frame $\{e_1, e_2, e_3\}$ is adapted to the Riemannian submersion π.*

We use Proposition 3.2 to prove the following important consequence which is used proving our main theorem

Theorem 3.3. *Let $\pi : S^3_\varepsilon \to (N^2, h)$ be a Riemannian submersion from Berger 3-sphere with $R = 4 - 4\varepsilon^2 \neq 0$. Then, we can choose an adapted frame is of the form $\{e_1 = a_1^1E_1 + a_1^2E_2,\ e_2,\ e_3\}$ to the Riemannian submersion π with e_3 being vertical. Moreover, if E_3 is not vertical, then $\nabla_{e_2}e_2 \neq 0$, i.e., $f_2 \neq 0$.*

Proof. It is observed that if the vertical vector field E_3 is tangent to the fiber of the Riemannian submersion π, then any basic vector field is of the form $e = a^2E_1 + b^2E_2$, and $a^2 + b^2 = 1$.

From this time now, we just need to suppose that the vertical vector field e_3 is not parallel to E_3. Then, the vector filed $e_1 = e_3 \times E_3$ is horizontal and hence $\langle e_1, E_3\rangle = 0$. From this, a defined orthonormal frame $\{e_1,\ e_2 = e_3\times e_1,\ e_3\}$ on M^3 is obtained. If denoting by $e_i = \sum\limits_{j=1}^{3} a_i^jE_j, i = 1, 2, 3$, together with $\langle e_1, E_3\rangle = 0$, then the vector horizontal filed e_1 is of the form $e_1 = a_1^1E_1 + a_1^2E_2$ and hence $(a_1^1)^2 + (a_1^2)^2 = 1$. From these, we have the following

$$a_1^3 = 0,\ a_3^3 \neq \pm1 \text{ and } a_2^3 \neq 0. \tag{34}$$

Moreover, one can also get the following equalities as

$$f_1 = 0,\ \nabla_{e_1}e_1 = 0. \tag{35}$$

Indeed, we compute

$$\nabla_{e_1}e_1 = \nabla_{e_1}(\sum_{i=1}^{3} a_1^iE_i) = \sum_{i=1}^{3} e_1(a_1^i)E_i + \sum_{i,j=1}^{3} a_1^ja_1^i\nabla_{E_j}E_i. \tag{36}$$

In addition, one uses (27) to see that the above has another expression as

$$\nabla_{e_1}e_1 = -f_1 e_2 = -f_1\sum_{i=1}^{3} a_2^i E_i. \tag{37}$$

We equate (36) and (37) and compare the coefficient of E_3 to obatin

$$\begin{aligned} &-f_1 a_2^3 = \langle -f_1 \sum_{i=1}^{3} a_1^i E_i, E_3\rangle = \langle \nabla_{e_1}e_1, E_3\rangle \\ &= \langle \sum_{i=1}^{3} e_1(a_1^i)E_i + \sum_{i,j=1}^{3} a_1^j a_1^i \nabla_{E_j}E_i, E_3\rangle = e_1(a_1^3) = 0, \end{aligned} \tag{38}$$

which has been used (9) and $a_1^3 = 0$. This follows $f_1 = 0$ for $a_2^3 \neq 0$, from which we get (35).

Using (9), (27) and $a_1^3 = f_1 = 0$, a further calculation analogous to those used computing (36)–(38) yields

$$\begin{cases} e_1(a_2^3) = -(\sigma+\varepsilon)a_3^3, \\ e_1(a_3^3) = (\sigma+\varepsilon)a_2^3, \\ e_2(a_2^3) = 0, \\ e_2(a_3^3) = 0, \\ e_3(a_2^3) = -\kappa_2 a_3^3, \\ e_3(a_3^3) = \kappa_2 a_2^3, \\ \kappa_1 a_3^3 = (\sigma - \varepsilon - f_3)a_2^3, \\ f_2 a_2^3 = (\sigma+\varepsilon)a_3^3. \end{cases} \tag{39}$$

Since $\nabla_{e_1}e_1 = 0$, one concludes from Proposition 3.2 to have either $\nabla_{e_2}e_2 \neq 0$, and the frame $\{e_1, e_2, e_3\}$ is an adapted to the Riemannian submersion π, or $\nabla_{e_2}e_2 = 0$. Now, we just need to consider the latter case $\nabla_{e_2}e_2 = 0$, i.e., $f_2 = 0$. Combining these, one has the following

$$a_1^3 = f_1 = f_2 = 0,\ a_3^3 \neq \pm 1 \text{ and } a_2^3 \neq 0. \tag{40}$$

We now show that the above case (i.e., $a_1^3 = f_1 = f_2 = 0$, $a_3^3 \neq \pm 1$, $a_2^3 \neq 0$) can not happen by considering the following two cases:

Case I: $a_3^3 = 0$ and $a_1^3 = f_1 = f_2 = 0$. One shows that the case can not happen.

In this case, since $a_1^3 = 0$, we have $a_2^3 = \pm 1$. We substitute $a_2^3 = \pm 1$ and $a_3^3 = 0$ into the 2nd equation of (39) separately to obtain $\sigma = -\varepsilon$ and hence

$f_3 = \sigma - \varepsilon = -2\varepsilon$. Substituting these and $f_1 = f_2 = 0$ into the 1st and the 2nd equation of (28), we get $\kappa_1 = \kappa_2 = 0$. From these and using the 2nd equation of (29), we get $a_2^3 = 0$, a contradiction.

Case II: $a_3^3 \neq 0, \pm 1$, $a_2^3 \neq 0, \pm 1$ and $a_1^3 = f_1 = f_2 = 0$. We will show that the case can not happen, either.
In this case, substituting $f_2 = 0$ into the 8th equation of (39), we obtain $\sigma = -\varepsilon$. Then, we apply the 5th equation of (29), the 1st and the 2nd equation of (39) separately to obtain $\kappa_2 = 0$, $e_1(a_2^3) = e_1(a_3^3) = 0$, and hence $e_3(a_2^3) = e_3(a_3^3) = 0$ by using the 5th and the 6th equation of (39). Combing these and using the 3rd equation and the 4th equation of (39), we find a_2^3, a_3^3 to be constants. On the other hand, we must have $\kappa_1 \neq 0$. If otherwise, i.e., $\kappa_1 = 0$, we then substitute $a_1^3 = 0 = \kappa_1 = \kappa_2 = 0$ and $\sigma = -\varepsilon$ into the 2nd equation of (29) to $a_2^3 = 0$ and hence $a_3^3 = \pm 1$, a contradiction. Therefore, combining these and using the 1st and the 4th equation of (29), we deduce κ_1 and f_3 being constants. Substituting this and $f_1 = f_2 = 0$ into the 1st equation of the Jacobi identities (28), we deduce $0 = e_1(f_3) = \kappa_1 f_3$ meaning $f_3 = 0$. However, using the 4th equation of (29), $\sigma = -\varepsilon \neq 0$ and $f_1 = f_2 = f_3 = 0$, one finds $(a_3^3)^2 = \frac{-4\varepsilon^2}{R} = \frac{4\varepsilon^2}{4\varepsilon^2 - 4}$ implying $4\varepsilon^2 - 4 > 0$ and hence $(a_3^3)^2 > 1$, a contradiction.
Combining Case I and Case II, the case $a_1^3 = f_1 = f_2 = 0$ and $a_3^3 \neq \pm 1$ can not happen.
Summarizing all results, we obtain the theorem.

□

Remark 2. Let $\pi : S_\varepsilon^3 \to (N^2, h)$ be a Riemannian submersion from a Berger 3-sphere with e_3 being vertical. Then, we can conclude the following facts:

(a): If $R = 4 - 4\varepsilon^2 = 0$, then the Berger 3-sphere is a standard sphere S^3. It is a fact from Theorem 3.3 in [33] that biharmonic Riemannian submersion $\pi : S^3 \to (N^2, h)$ has to be harmonic. Actually, the biharmonic Riemannian submersion can be expressed as the Riemannian submersion $\pi : S^3 \to S^2(\frac{1}{2})$, $\psi(z, w) = \frac{1}{2}(2zw, |z|^2 - |w|^2)$, where $z = x^1 + ix^2$, $w = x^3 + ix^4$, up to equivalence.

(b): If $a_3^3 = \pm 1$, i.e., the vertical vector field e_3 is parallel to E_3, then it is not difficult to see from (9) that the tension of the Riemannian submersion $\tau(\pi) = -d\pi(\nabla^M_{E_3} E_3) = 0$, i.e., π is harmonic. Moreover, the biharmonic Riemannian submersion can be represented as the Riemannian submersion $\pi : S_\varepsilon^3 \to S^2(\frac{1}{2})$, $\psi(z, w) = \frac{1}{2}(2zw, |z|^2 - |w|^2)$, where $z = x^1 + ix^2$, $w = x^3 + ix^4$, up to equivalence.

Remark 3. Let $\pi : S^3_\varepsilon \to (N^2, h)$ be a Riemannian submersion from a Berger 3-sphere with e_3 being vertical. If E_3 is not vertical (i.e., $a^3_3 \neq \pm 1$) and $R = 4 - 4\varepsilon^2 \neq 0$, then it follows from Theorem 3.3 that there exists such an orthonormal frame $\{e_1 = a^1_1 E_1 + a^2_1 E_2, e_2, e_3\}$ adapted to the Riemannian submersion π, and $a^3_1 = f_1 = f_3 = 0$, $f_2 \neq 0$ and $a^3_2 \neq 0$.

Now, we will prove our main results as follows

Theorem 3.4. *A Riemannian submersion $\pi : S^3_\varepsilon \to (N^2, h)$ from a Berger 3-sphere to a surface is biharmonic if and only if it is harmonic.*

Proof. Let ∇ denote the Levi-Civita connection on S^3_ε and by $e_i = \sum_{j=1}^{3} a^j_i E_j$, $i = 1, 2, 3$. To obtain the theorem, by Remark 2, we just need to consider the case $R = 4 - 4\varepsilon^2 \neq 0$ and $a^3_3 \neq \pm 1$. Therefore, by Theorem 3.3 and Remark 3, we can take an adapted frame as the form $\{e_1 = a^1_1 E_1 + a^2_1 E_2,\ e_2,\ e_3\}$ to the Riemannian submersion π with e_3 being vertical, and together with a result in [33], we have

$$a^3_2,\ a^3_3 \neq 0, \pm 1,\ a^3_1 = f_1 = f_3 = 0,\ e_3(f_1) = e_3(f_2) = 0,\ f_2 \neq 0. \tag{41}$$

We will show that the above case can not happen by the following two steps:
Step 1: Show that $e_2(f_2) = e_2(\kappa_1) = e_3(\kappa_1) = e_2(\sigma) = e_3(\sigma) = \kappa_2 = 0,\ \kappa_1 \neq 0,$ and $\sigma \neq 0, \pm\varepsilon$.

Firstly, show that $\sigma \neq 0$. Clearly, if $\sigma = 0$, we use the 1st equation of (29) to have $a^3_2 a^3_3 = 0$ contradicting (41). This leads to $\sigma \neq 0$.

Secondly, show that $\kappa_2 = e_2(\kappa_1) = e_2(\sigma) = e_3(\sigma) = 0$.
A straightforward calculation using the 5th , the 6th equation of (39), (31) and (41) and applying e_3 to both sides of the 4th equation of (29) and the 8th equation of (39) separately, we get

$$e_3(\sigma) = -\tfrac{1}{3\sigma}\kappa_2 a^3_2 a^3_3 R, \tag{42}$$

and

$$-f_2\kappa_2 a^3_3 = e_3(\sigma) a^3_3 + (\sigma + \varepsilon)\kappa_2 a^3_2. \tag{43}$$

We substitute (42) into (43) and simplify the resulting equation to obtain

$$\kappa_2\left(3f_2\sigma a^3_3 - a^3_2 (a^3_3)^2 R + 3\sigma(\sigma + \varepsilon) a^3_2\right) = 0, \tag{44}$$

which means $\kappa_2 = 0$, or,

$$3f_2\sigma a^3_3 - a^3_2 (a^3_3)^2 R + 3\sigma(\sigma + \varepsilon) a^3_2 = 0. \tag{45}$$

Together with (41), substituting the 8th equation of (39) into the above equation and simplifying the resulting equation, we have

$$3\sigma(\sigma+\varepsilon)=(a_2^3a_3^3)^2R. \tag{46}$$

Noting that $f_3=0$, we rewritten the 8th equation of (39) as

$$\kappa_1 a_3^3=(\sigma-\varepsilon)a_2^3. \tag{47}$$

Using the 3rd, the 4th equation of (39) and (41) and applying e_2 to both sides of the above equation gives

$$3(2\sigma+\varepsilon)e_2(\sigma)=0. \tag{48}$$

This implies $e_2(\sigma)=0$.

From these and using the 5th equation of (29) and (42), we have $\kappa_2=0$ and hence $e_3(\sigma)=e_2(\kappa_1)=0$.

Thirdly, show that $e_2(f_2)=e_3(\kappa_1)=0$.

Using the 3rd equation of (39), (41) and applying e_2 to both sides of the 2nd equation of (29) and the 8th equation of (39) separately, together with $e_2(\kappa_1)=e_2(\sigma)=0$, we get

$$e_2e_1(\kappa_1)=0,\ e_2(f_2)=0. \tag{49}$$

From these, we get $e_1e_2(\kappa_1)-e_2e_1(\kappa_1)=0$, which, together with $e_1e_2(\kappa_1)-e_2e_1(\kappa_1)=[e_1,e_2](\kappa_1)=-2\sigma e_3(\kappa_1)$, implies $e_3(\kappa_1)=0$.

Finally, show that $\sigma\neq\pm\varepsilon$ and $\kappa_1\neq 0$.
Using (41) and $\kappa_2=0$, the 7th equation of (29) becomes

$$\kappa_1 f_2=\sigma^2-\varepsilon^2. \tag{50}$$

Since $f_2\neq 0$, one sees from (50) that $\sigma=\pm\varepsilon$ is equivalent to $\kappa_1=0$. Obviously, if $\sigma=\pm\varepsilon$ and hence $\kappa_1=0$, by the 2nd equation of (29) and (41), one sees $R=0$, a contradiction. Therefore, we get $\sigma\neq\pm\varepsilon$ and $\kappa_1\neq 0$.

Step 2: show that $\sigma=-\varepsilon$, a contradiction.
For $f_1=\kappa_2=0$ and $K^N=e_1(f_2)-f_2^2$, it is easy to deduce that biharmnic equation (33) reduces to

$$\Delta\kappa_1-\kappa_1\{-e_1(f_2)+2f_2^2\}=0. \tag{51}$$

Together with (41), using the 1st equation of (39) and applying e_1 to both sides of the 2nd equation of (29) and (50) separately, we obtain

$$\begin{cases} e_1e_1(\kappa_1) = 2\kappa_1 e_1(\kappa_1) - 2\sigma e_1(\sigma) - 2(\sigma+\varepsilon)a_2^2 a_3^3 R, \\ \kappa_1 e_1(f_2) + f_2 e_1(\kappa_1) = 2\sigma e_1(\sigma). \end{cases} \tag{52}$$

We substitute the 1st, the 2nd equation of (29), the 8th equation of (39), (52), the results of Step 1 into biharmnic equation (51), together with $f_1 = \kappa_2 = 0$ and (41), to have

$$\kappa_1^3 - 3\kappa_1^2 f_2 + \kappa_1 (a_2^3)^2 R - 4(\sigma+\varepsilon)a_2^3 a_3^3 R = 0. \tag{53}$$

Multiplying a_3^3 to both sides of the above equation and using the fact that $\kappa_1 a_3^3 = (\sigma-\varepsilon)a_2^3$, $\kappa_1 f_2 = \sigma^2 - \varepsilon^2$ and $(a_2^3)^2 + (a_3^3)^2 = 1$ and simplifying the resulting equation we get

$$\kappa_1^2 = \tfrac{5\sigma+3\varepsilon}{\sigma-\varepsilon} R (a_3^3)^2 + 3\sigma^2 - 3\varepsilon^2 - R. \tag{54}$$

We substitute $\kappa_1 a_3^3 = (\sigma-\varepsilon)a_2^3$ into the above equation and simplify the resulting equation to obtain

$$\tfrac{5\sigma+3\varepsilon}{\sigma-\varepsilon} R (a_3^3)^4 + (4\sigma^2 - 2\sigma\varepsilon - 2\varepsilon^2 + R)(a_3^3)^2 - (\sigma-\varepsilon)^2 = 0. \tag{55}$$

Applying e_1 to both sides of (54) and using the 1st equation and the 2nd equation of (29) to simplify the resulting equation we have

$$\kappa_1^3 = \kappa_1(7\sigma^2 - \varepsilon^2) - \kappa_1 (a_2^3)^2 R - \tfrac{8\kappa_1\sigma\varepsilon R}{(\sigma-\varepsilon)^2}(a_3^3)^2 - \tfrac{4\varepsilon R^2}{(\sigma-\varepsilon)^2} a_2^3 (a_3^3)^3 + \tfrac{(5\sigma+3\varepsilon)(\sigma+\varepsilon)}{\sigma-\varepsilon} R a_2^3 a_3^3 + 3\sigma a_2^3 a_3^3 R. \tag{56}$$

We multiply a_3^3 to both sides of the above equation and use the fact that $\kappa_1 a_3^3 = (\sigma-\varepsilon)a_2^3$ and $(a_2^3)^2 + (a_3^3)^2 = 1$ and simplify the resulting equation to get

$$\kappa_1^2 = -\tfrac{4\varepsilon R^2}{(\sigma-\varepsilon)^3}(a_3^3)^4 + \tfrac{(9\sigma^2 - 5\sigma\varepsilon + 4\varepsilon^2)R}{(\sigma-\varepsilon)^2}(a_3^3)^2 + 7\sigma^2 - \varepsilon^2 - R. \tag{57}$$

Comparing (54) with the above equation and simplifying the resulting equation we have

$$-\tfrac{4\varepsilon R^2}{(\sigma-\varepsilon)^3}(a_3^3)^4 + \tfrac{(4\sigma^2 - 3\sigma\varepsilon + 7\varepsilon^2)R}{(\sigma-\varepsilon)^2}(a_3^3)^2 + 2(2\sigma^2 + \varepsilon^2) = 0. \tag{58}$$

Eq.(58) multiplied by $(5\sigma+3\varepsilon)(\sigma-\varepsilon)^2$ minus Eq. (55) multiplied by $(-4\varepsilon R)$, a straightforward computation yields

$$\begin{aligned} &\left((5\sigma+3\varepsilon)(4\sigma^2 - 3\sigma\varepsilon + 7\varepsilon^2 + 4\varepsilon(4\sigma^2 - 2\sigma\varepsilon - 2\varepsilon^2 + R)\right) R (a_3^3)^2 \\ &= 2(\sigma-\varepsilon)^2 \left(-(5\sigma+3\varepsilon)(2\sigma^2+\varepsilon^2) + 2\varepsilon R\right). \end{aligned} \tag{59}$$

On the other hand, Eq.(58) multiplied by $(\sigma-\varepsilon)^2$ plus Eq. (55) multiplied by $2(2\sigma^2+\varepsilon^2)$, and we then simplify the resulting equation to obtain

$$\begin{aligned}&(\sigma-\varepsilon)\left((4\sigma^2-3\sigma\varepsilon+7\varepsilon^2)R-2(4\sigma^2-2\sigma\varepsilon-2\varepsilon^2+R)(2\sigma^2+\varepsilon^2)\right)\\&=2\left(-(5\sigma+3\varepsilon)(\sigma^2+\varepsilon^2)+2\varepsilon R\right)R(a_3^3)^2.\end{aligned}\tag{60}$$

A direct computation using Eq. (59) and Eq.(60) to simplify the resulting equation we get

$$\begin{aligned}&\left((5\sigma+3\varepsilon)(4\sigma^2-3\sigma\varepsilon+7\varepsilon^2)+2l(4\sigma^2-2\sigma\varepsilon-2\varepsilon^2+R)\right)\\&\times\left((4\sigma^2-3\sigma\varepsilon+7\varepsilon^2)R+2(4\sigma^2+2\sigma\varepsilon-2\varepsilon^2+R)(2\sigma^2+\varepsilon^2)\right)\\&=4(\sigma-\varepsilon)\left(-(5\sigma+3\varepsilon)(2\sigma^2+\varepsilon^2)+2\varepsilon R\right)^2\end{aligned}\tag{61}$$

A further computation, one finds that the above equation is a polynomial system in σ of degree seven with constant coefficients as

$$80\sigma^7+P(\sigma)=0,\tag{62}$$

where $P(\sigma)$ denotes a polynomial in σ of not more than 6 with constant coefficients. This implies that σ have to be constant. Using Eq.(58) and Eq.(57), we see that both a_3^3 and κ_1 are constants. Then $a_2^3=\pm\sqrt{1-(a_3^3)^2}$ has to be a constant since $a_1^3=0$. From these and using the 1st equation of (39), we obtain $\sigma=-\varepsilon$, which is a contradiction.
Summarizing the results proved above, the theorem follows. □

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